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# CONSISTENCY, STABILITY AND CONVERGENCE OF THE FINITE-DIFFERENCE EQUATIONS FOR FLOW ABOUT A ROTATING SPHERE IN AN AXIAL STREAM

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## SUMMARY

The finite-difference equations which have previously been developed to solve the problem of laminar boundary layer flow about a rotating sphere in an axial stream are analysed according to the available numerical stability theories. This analysis is necessary to determine the restrictions on velocities and mesh sizes required to obtain a convergent numerical solution. Convergence can be achieved if both consistency and stability of the finite-difference equations are fulfilled. The analysis reported in the present paper shows that the developed finite-difference equations are consistent with their original partial differential equations. Also, the analysis proves that the developed finite-difference procedure is numerically stable for all mesh sizes as long as the downstream meridional velocity is non-negative, i.e. as long as no flow reversals occur within the domain of solution.

KEY WORDS Finite-difference Stability Rotating Sphere in Stream

## INTRODUCTION

In approximating the derivatives of a mathematical model by numerical finite differences, there is an error introduced which is usually termed the truncation error. Moreover, when the numerical method is actually run on a digital computer, round-off errors are introduced. Round-off errors are caused by rounding of the results from individual arithmetic operations because only a finite number of digits can be retained, by the computer, after each operation.

Having constructed a plausible finite-difference procedure, we must check whether it will be convergent or not. Convergence means that the numerical solution of a finite-difference procedure tends to the exact solution of the original partial differential equation(s) as the grid spacing tends to zero.

There are two important concepts closely associated with the convergence of a particular finite-difference procedure, namely those of consistency and stability. Consistency means that the procedure may in fact approximate the solution of the partial differential equation(s) under study, and not the solution of some other differential equation(s). To obtain this requirement, the

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truncation errors must tend to zero as the numerical mesh size tends to zero. The term stability is used in reference to the behaviour of the round-off errors or any other errors whether present in the initial conditions, or brought in via the boundary conditions, or arising from any sort of error in the calculations. It is said that a finite-difference scheme is numerically stable if such errors introduced at any stage of computation either decreases or remains constant during subsequent uninterrupted stepping ahead of the solution. According to a theorem due to Lax,[1] for linear finite-difference equations, provided that consistency is satisfied, stability is both a necessary and sufficient condition for convergence. More details concerning consistency and stability and theories relating them with convergence may be found in Reference 2.

In a recent publication, El-Shaarawi *et al.*[3] presented a simple non-iterative finite-difference scheme to solve the problem of steady laminar boundary layer flow about a rotating sphere in an axial stream. The main aims of the present paper are to check whether the finite-difference equations are consistent with their original differential equations and to make an analysis to determine the restrictions on velocities and mesh sizes required to obtain a convergent numerical solution.

## GOVERNING EQUATIONS AND THEIR FINITE DIFFERENCE REPRESENTATIONS

Consider steady, rotationally symmetric, laminar flow of an incompressible Newtonian fluid with constant physical properties in the region outside a sphere which is rotating with a constant angular velocity $\Omega$ about a diameter parallel to the flow direction. Let $x, y, z$ be orthogonal curvilinear co-ordinates, where $x$ is measured along a meridional direction, $y$ is along a circular cross-section of the sphere by a plane perpendicular to the axis of rotation, and $z$ is in the spherical radial direction with its zero value located on the sphere surface. Under these previously mentioned assumptions and in the absence of body forces, the dimensionless boundary layer equations and boundary conditions for the problem at hand are

$$\frac{\partial U}{\partial X} + \frac{U}{R}\frac{\mathrm{d}R}{\mathrm{d}X} + \frac{Re}{2}\frac{\partial W}{\partial Z} + Re\frac{W}{1+z} = 0, \tag{1}$$

$$U\frac{\partial U}{\partial X} - \frac{Ta}{Re^2}\frac{V^2}{R}\frac{\mathrm{d}R}{\mathrm{d}X} + \frac{Re}{2}W\frac{\partial U}{\partial Z} = U_0^*\frac{\mathrm{d}U_0^*}{\mathrm{d}X} + \frac{\partial^2 U}{\partial Z^2}, \tag{2}$$

$$U\frac{\partial V}{\partial X} + \frac{UV}{R}\frac{\mathrm{d}R}{\mathrm{d}X} + \frac{Re}{2}W\frac{\partial V}{\partial Z} = \frac{\partial^2 V}{\partial Z^2}, \tag{3}$$

$$\left.\begin{aligned}
&\text{for } Z=0 \text{ and } X>0: U=W=0 \text{ and } V=\sin\theta,\\
&\text{for } Z\geqslant\delta^* \text{ and } X>0: U=U^*=\left[1+\frac{1}{2(1+Z)^3}\right]\sin\theta, \text{ and } V=0,\\
&\text{for } X=0 \text{ and } Z\geqslant 0: U=V=0, \text{ and } W=-\left[1+\frac{1}{(1+Z)^3}\right].
\end{aligned}\right\} \tag{4}$$

In the above equations $X=2x/(a.Re)$, $Z=z/a$, $R=2r/(a.Re)$, $U=u/U_\infty$, $W=w/U_\infty$, $V=v/\Omega a$, $Re=2a\,U_\infty/\nu$, $Ta=4\Omega^2a^4/\nu^2$, $U^*=u^*/U_\infty$ and $\delta^*=\delta/a$, where $a$ is the radius of the sphere, $U_\infty$ is the free stream velocity, $\nu$ is the kinematic viscosity of the fluid, $u, v$ and $w$ are the meridional, azimuthal and radial components of velocity, respectively, $r$ is the radius of a circular cross-section of the sphere by a plane perpendicular to the axis of rotation, and the subscript 0 denotes 'on the sphere surface'.

The finite-difference equations which have been used by EL-Shaarawi *et al.* are repeated here for completeness:

$$\frac{U_{i+1,j+1}+U_{i,j+1}-U_{i+1,j}-U_{i,j}}{2\Delta X_{i+1/2}}+\frac{(U_{i+1,j+1}+U_{i,j+1})Re}{4(1+Z_{i+1/2})\cot(j\Delta\theta)}$$
$$+\frac{Re}{2}\frac{W_{i+1,j+1}-W_{i,j+1}}{\Delta Z}+\frac{Re(W_{i+1,j+1}+W_{i,j+1})}{2(1+Z_{i+1/2})}=0, \quad (5)$$

$$U_{i,j}\frac{U_{i,j+1}-U_{i,j}}{\Delta X_i}-\left[\frac{Ta}{Re}\right]\frac{V_{i,j}V_{i,j+1}}{2[1+(i-1)]\Delta Z}\cot(j\Delta\theta)$$
$$+W_{i,j}\frac{Re}{2}\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2\Delta Z}=\tfrac{3}{2}\sin(j\Delta\theta)\tfrac{3}{2}Re\cos(j\,\Delta\theta)$$
$$+\frac{U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}}{(\Delta Z)^2}, \quad (6)$$

$$U_{i,j}\frac{V_{i,j+1}-V_{i,j}}{\Delta X_i}+\frac{Re}{4}U_{i,j}\frac{V_{i,j+1}+V_{i,j}}{1+(i-1)\Delta Z}\cot[(j-1/2)\Delta\theta]$$
$$+W_{i,j}\frac{V_{i+1,j+1}+V_{i+1,j}-V_{i-1,j}-V_{i-1,j+1}}{4\Delta Z}\frac{Re}{2}$$
$$=\frac{V_{i+1,j+1}+V_{i+1,j}-2V_{i,j+1}-2V_{i,j}+V_{i,j+1}+V_{i-1,j}}{2(\Delta Z)^2}, \quad (7)$$

$$U_{i,1}=V_{i,1}=0,$$
$$W_{i,1}=\frac{1}{[1+(i-1)\Delta Z]^3}-1 \quad (8)$$
$$U_{1,j}=W_{1,j}=0$$
$$V_{1,j}=\sin[(j-1)\Delta\theta]$$
$$V_{n+1,j}=0,$$
$$U_{n+1,j}=\left[1+\frac{1}{2(1+n\Delta Z)^3}\right]\sin[(j-1)\Delta\theta].$$

Figure 1 shows the numerical grid, where the independent variables are computed at the intersections of the grid lines and $(i,j)$ is a typical mesh point. Mesh points are numbered consecutively, the $i$ is progressing in the radial direction with $i=1,2,3,\ldots,n+1$ from the sphere surface, and the $j$ is progressing in the meridional direction with $j=1,2,3,\ldots,m+1$ from the stagnation point. At each meridional station $j$, the number of radial increments $n$ should be chosen so that the uppermost point ($i=n+1$) lies in essentially undisturbed fluid. In Figure 2 parts of the finite-difference domain are drawn to clarify how each differential equation has been transferred into its corresponding finite-difference form. In this Figure, the crossed points represent these grid points involved in the finite-difference representation of the equation under consideration and a ringed point is that point at which the derivatives have been calculated.

As can be seen, the finite difference approximations are not perfectly symmetrical nor are they of the same form in all equations; this was done to ensure numerical stability, to enable the equations

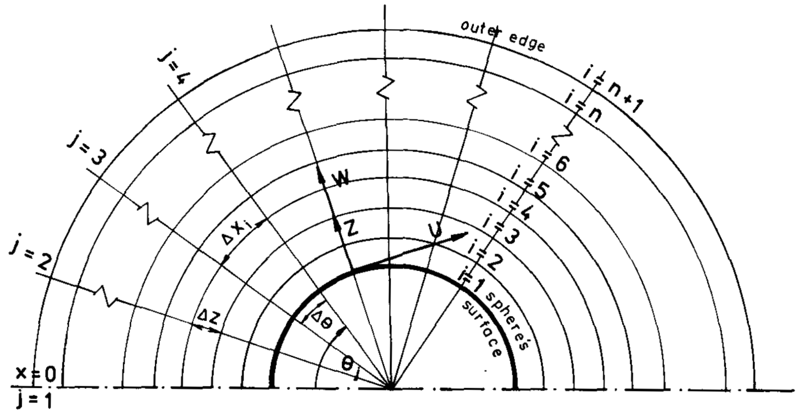


Figure 1. Numerical grid

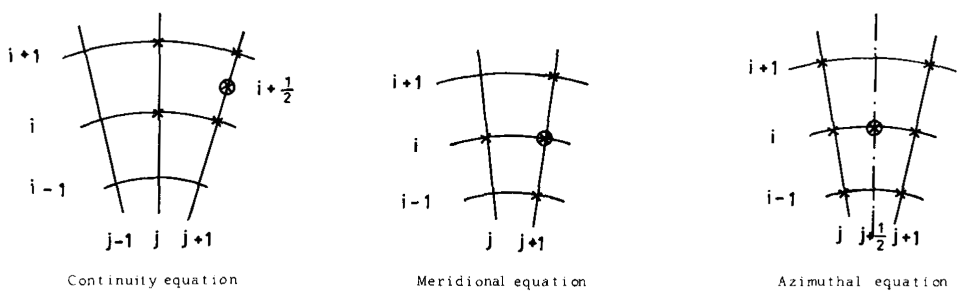


Figure 2. Grid points involved in difference representations of the three differential equations

to be uncoupled from each other and to be solved by the method summarized in the paper of El-Shaarawi *et al.*[3]

The truncation error due to the approximation of any of equations (1)–(3) by its corresponding finite-difference equation is proportional to $(\Delta X)$ and to $(\Delta Z)^2$. Such truncation errors vanish as the mesh sizes tend to zero, and hence, the finite-difference equations (5)–(7) are consistent representations of equations (1)–(3).

## STABILITY ANALYSIS

The finite-difference equations (5)–(7) have been linearized by assuming that, where the product of two unknowns (with subscript $j+1$) occurs, one of them is given approximately by its known value at the previous meridional step (with subscript $j$). This means that, throughout a meridional step, the non-linear coefficients have been frozen locally at values ($U$, $V$ and $W$). For such a system, i.e. equations (5)–(7), one may apply linear stability theory as follows.

According to von Neumann analysis (summarized by Carnahan *et al.*[2]) the numerical stability of a finite-difference procedure can be examined by introducing small perturbations (denoted hereinafter by $U'$, $V'$ and $W'$) into the finite-difference equations and checking whether or not such perturbations amplify as the computation proceeds in the marching direction. Therefore, the insertion of the new variables $U+U'$, $V+V'$ and $W+W'$ into equations (5)–(7) leads to the following three equations:

$$\frac{U'_{i+1,j+1}+U'_{i,j+1}-U'_{i+1,j}-U'_{i,j}}{2\Delta X_{i+1/2}}+\frac{(U'_{i+1,j+1}+U'_{i,j+1})\cot\theta|_{j+1}}{2(1+Z_i)}\frac{Re}{2}$$
$$+\frac{(W'_{i+1,j+1}+W'_{i,j+1})}{(1+Z_{i+1})+(1+Z_i)}\frac{Re}{2}+\frac{W'_{i+1,j+1}-W'_{i,j+1}}{\Delta Z}\frac{Re}{2}=0, \tag{9}$$

$$U\frac{U'_{i,j+1}-U'_{i,j}}{\Delta X}-\frac{Ta}{Re^2}V\frac{V'_{i,j+1}\cot\theta|_{j+1}}{(1+Z_i)}\frac{Re}{2}+W\frac{U'_{i+1,j+1}-U'_{i+1,j+1}}{2\Delta Z}\frac{Re}{2}$$
$$=\frac{U'_{i+1,j+1}-2U'_{i,j+1}+U'_{i-1,j+1}}{(\Delta Z)^2} \tag{10}$$

$$U\frac{V'_{i,j+1}-V'_{i,j}}{\Delta X_i}+U\frac{Re}{2}\frac{(V'_{i,j+1}+V'_{i,j})\cot\theta|_{j+1/2}}{2(1+Z_i)}+W\frac{V'_{i+1,j+1}+V'_{i+1,i}-V'_{i-1,j}-V'_{i-1,j+1}}{4(\Delta Z)}\frac{Re}{2}$$
$$=\frac{V'_{i+1,j+1}+V'_{i+1,j}-2V'_{i,j+1}-2V'_{i,j}+V'_{i-1,j+1}+V'_{i-1,j}}{2(\Delta Z)^2} \tag{11}$$

Equations (9)–(11) govern the behaviour of the small perturbations which represent the round-off or similar errors. According to von Neumann, a general term of a particular numerical error (i/e. a primed variable) at a point $(X, Z)$ is a product of two functions; the first of these is an exponential function of $X$ only, which represents the amplitude of the error at the particular point under consideration, and the second is an exponential function of $Z$ only, containing all possible existing harmonics. A typical form of this general term at any station (say for example $j$) is $f(X)e^{iqZ}$, where i denotes the square root of $-1$, $q$ is any real number representing the frequency of any existing harmonic, and $f(X)$ is an exponential function of $X$ only, representing the amplitude of the error at that particular station.

Now using such a typical form for all the perturbations, we have $U_{i,j}=f_1(X)e^{iqZ}$, $V_{i,j}=f_2(X)e^{iqZ}$ and $W_{i,j}=f_3(X)e^{iqZ}$. Substituting these sinusoidal representations of the perturbations into equations (9)–(11), leads, after manipulation, to the following set of three simultaneous equations:

$$\begin{aligned} f_1(X+\Delta X)&=C_1f_1(X)+c_2f_2(x),\\ f_2(X+\Delta X)&=C_3f_2(X),\\ f_3(X+\Delta X)&=C_4f_1(X)+C_5f_2(X), \end{aligned} \tag{12}$$

in which

$$\begin{aligned} C_1&=1/[1+4S\sin^2\gamma+iSB],\\ S&=\Delta X/[U(\Delta Z)^2],\\ \gamma&=q\Delta Z/2,\\ B&=W\Delta Z\,Re\sin(q\Delta Z)/2,\\ C_2&=C_3C_1V\Delta X\,Ta\cot\theta_{j+1}/[Re(1+Z_i)U],\\ C_3&=[1-Q-2S\sin^2\gamma+(1/4)iSB]/[1+Q+2S\sin^2\gamma-(1/4)iSB],\\ Q&=Re\cot\theta_{j+1/2}/[4\Delta X(1+Z_i)], \end{aligned}$$

$$C_4 = \left[\frac{1}{Re\,\Delta X_{i+1/2}} - C_1\left(\frac{1}{Re\,\Delta X_{i+1/2}} + \frac{\cos\theta_{j+1}}{2(1+Z_{i+1/2})}\right)\right] \Big/ \left[\frac{1}{1+Z_{i+1/2}} + \frac{e^{iq\Delta Z}-1}{e^{iq\Delta Z}+1}\right],$$

$$C_5 = -C_2\left[\frac{\cos\theta_{j+1}}{2(1+Z_i)} - \frac{1}{Re\,Z_{i+1/2}}\right] \Big/ \left[\frac{1}{1+Z_{i+1/2}} + \frac{e^{iq\Delta Z}-1}{\Delta Z(e^{iq\Delta Z}+1)}\right]$$

The foregoing set of equations (12) can be written in the following matrix–vector notations:

$$\mathbf{F}(X+\Delta X) = \mathbf{G}\mathbf{F}(X),$$

where $\mathbf{F}(X+\Delta X)$ and $\mathbf{F}(X)$ are the column vectors whose components are the amplitudes of the individual error components corresponding to the various dependent variables at $(X+\Delta X)$ and $X$, respectively, and $\mathbf{G}$ is a complex matrix of dimensions $3 \times 3$, known as the amplification matrix.

For numerical stability, each eigenvalue of the amplification matrix $\mathbf{G}$ must not exceed unity in modulus for all values of the frequency $q$. The eigenvalues $L$ of the amplification matrix $\mathbf{G}$ can be obtained by solving the equation

$$|\mathbf{G} - L\mathbf{I}^*| = 0,$$

where $\mathbf{I}^*$ is the identity matrix and $|\mathbf{G} - L\mathbf{I}^*|$ is the determinant which results from subtraction of the matrix $L\mathbf{I}^*$ from the amplification matrix $\mathbf{G}$. The solution of this equation gives

$$L_1 = C_1, \quad L_2 = C_3 \text{ and } L_3 = 0,$$

where $L_1, L_2$ and $L_3$ are the three eigenvalues of $\mathbf{G}$.

Denoting the moduli of these eigenvalues by $L_1', L_2'$ and $L_3'$, respectively, then

$$L_1' = 1/\sqrt{[(1+4S\sin^2\gamma)^2 + S^2B^2]},$$

$$L_2' = \sqrt{[(1-Q-2S\sin^2\gamma)^2 + S^2B^2]}/\sqrt{[(1+Q+2S\sin^2\gamma)^2 + S^2B^2]},$$

$$L_3' = 0$$

In the above expressions it is to be noted that the squared quantities are always positive, and hence each of the moduli $L_1'$ and $L_2'$ will be always less than unity if the variable $S$ is positive. However, $S$ is positive if $U$ is positive. Hence, the finite difference equations (5)–(7) are stable for all mesh sizes as long as the downstream meridional velocity $U$ is positive, i.e. as long as no flow reversals occur within the domain of solution.

## CONCLUSIONS

It has been proved that the finite-difference equations (5)–(7) are consistent with their original partial differential equations (1)–(3) and that they are also stable for all mesh sizes as long as the downstream meridional velocity component is non-negative. Thus, both the consistency and stability conditions have been fulfilled, on condition that no flow reversals occur within the domain of solution. Therefore, as long as the condition of absence of flow reversals is satisfied, the numerical solution of the finite difference equations (5)–(7) is convergent, i.e. it tends to the exact solution of the original governing partial differential equations (1)–(3) as the grid spacings tend to zero.

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